Berkeley mini Math Tournament 2024

Relay Round

April 14, 2024

Time limit: 40 minutes.

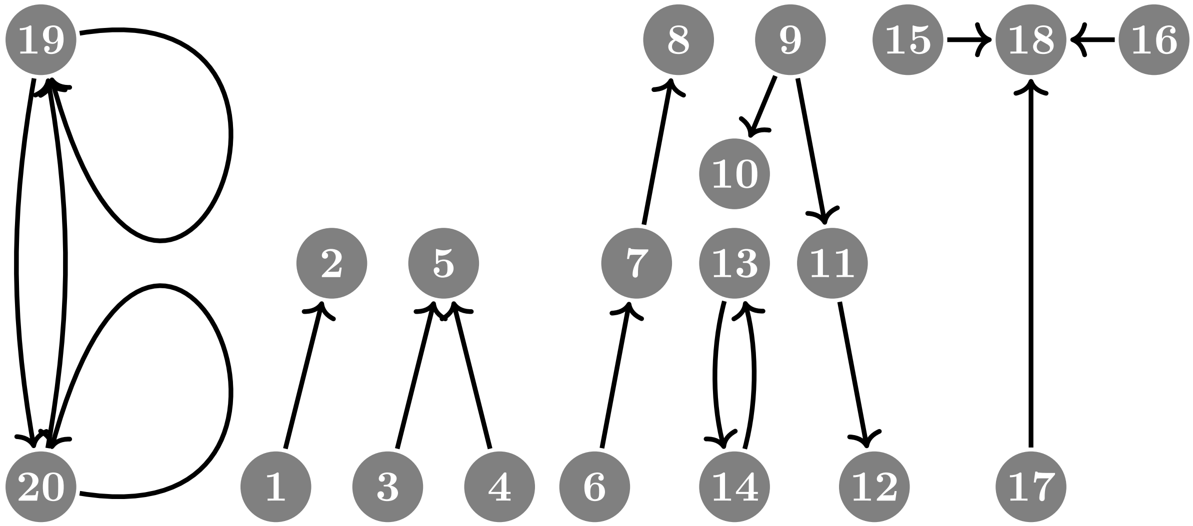
Instructions: For this test, you will work in teams of up to five to solve 7 sets of short answer questions, with 20 questions in total. You may work on any problem from any set at any time. Submit a single answer sheet for grading. Only answers written inside the boxes on the answer sheet will be considered for grading.

Note: Some questions may depend on other questions' answers. The drawing at the top of every set indicates which other answers are used for which problems. The integrated drawing of all 7 sets will be shown on the top of the first page. All answers for this round are numerical values (not in terms of any variables).

No calculators.

Answer format overview:

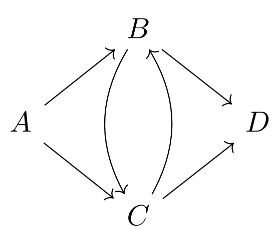
- Carry out any reasonable calculations. For instance, you should evaluate $\frac{1}{2}+\frac{1}{3}$, but you do not need to evaluate large powers such as 7^8.
- Write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. You may use constants such as π in your answers.
- Move all square factors outside radicals. For example, write $3\sqrt{7}$ instead of $\sqrt{63}$.
- Denominators do *not* need to be rationalized. Both $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$ are acceptable.
- Do not express an answer using a repeated sum or product.



Set 1

$$1 \longrightarrow 2$$

1. Aditya is walking in a park. There are four statues in the park and one-way roads between some pairs of statues, as shown by the letters and arrows in the diagram below. Aditya may visit the same statue more than once, but he refuses to travel along the same road more than once. In how many ways can he travel from statue A to statue D?

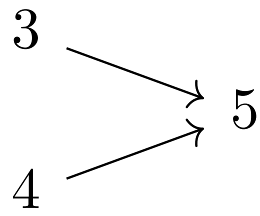


2. Let N_1 be the answer to Problem 1.

   What is the sum of all positive integers x such that

$$x + N_1 = x^{N_1/x}?$$

Set 2



3. Suppose a, b, c, and m are numbers satisfying the system of equations:

$$\frac{a+b+c}{3} = m,$$
$$a - m = -20,$$
$$b - m = -24.$$

What is the value of $c - m$?

4. A bag contains 9 cinnamon candies and 1 cherry candy. Nikki takes five candies from the bag uniformly at random without replacement. What is the probability that the second candy she takes is a cherry candy?

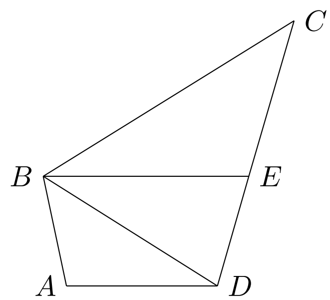
5. Let N_3 be the **units digit** of the answer to Problem 3 and N_4 be the answer to Problem 4.

Luke the frog and Barti the dolphin are racing across the length of a pond, starting at the same time. Luke hops at the speed of N_3 meters per second, and Barti swims at the speed of $\frac{1}{N_4}$ meters per second. Given Luke reaches the other side of the pond 30 seconds after Barti, what is the length of the pond, in meters?

Set 3

$$6 \longrightarrow 7 \longrightarrow 8$$

6. In quadrilateral $ABCD$, point E lies on $\overline{CD}$ such that angle $\angle EBC = 32°$, angle $\angle BCE = 40°$, $BD = BE$, and $\overline{AD}$ and $\overline{BE}$ are parallel. Compute angle $\angle ADB$ in degrees. Note that the diagram below is not drawn to scale.



7. Let N_6 be the answer to Problem 6.

   Define a function $f(x) = \frac{x^2}{1+x^2}$. Let

   $$A = f(1) + f(2) + f(3) + \cdots + f(N_6)$$

   and

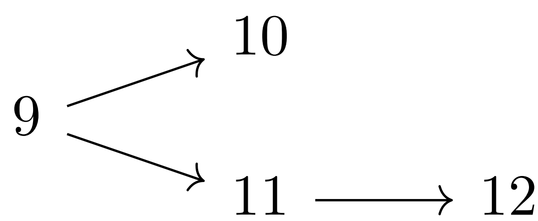
   $$B = f(1) + f\left(\frac{1}{2}\right) + f\left(\frac{1}{3}\right) + \cdots + f\left(\frac{1}{N_6}\right).$$

   Find $A + B$.

8. Let N_7 be the answer to Problem 7.

   Theo draws rectangle $ABCD$ in the plane. He makes a copy of this rectangle in the same plane, rescales the copy by some factor, and shifts the copy in the plane to produce a similar rectangle, $A_1B_1C_1D_1$, such that $\overline{AB}$ is parallel to $\overline{A_1B_1}$. Given $AA_1 = \sqrt{65}$, $BB_1 = \sqrt{20}$, and $CC_1 = \sqrt{N_7}$, compute DD_1.

Set 4



9. Nine cupcakes are arranged in a line in increasing size from left to right. In the line, the cupcake flavors alternate between chocolate and vanilla with the leftmost cupcake being chocolate. Andrew eats one chocolate cupcake, one vanilla cupcake, and then another chocolate cupcake, so that each cupcake eaten is larger than all cupcakes eaten before it. In how many ways can Andrew choose his three cupcakes?

10. Let N_9 be the answer to Problem 9.

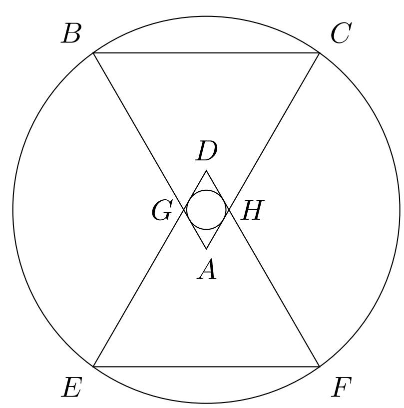
    Clara plays a game with a list of the integers from 1 to N_9, inclusive. First, Clara randomizes the order of the list. Then, in each round of the game, she removes the first and last elements from the list and adds the bigger of these numbers to her score. If she began the game with a score of 0, what is the smallest possible score that Clara could have achieved at the end of the game?

11. Let N_9 be the answer to Problem 9.

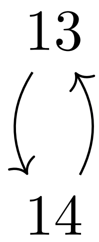
    A sequence of real numbers $a_0, a_1, a_2, \ldots$ satisfies $a_0 = N_9$ and $a_{n+1} = a_n - a_{n-1}$ for $n \geq 1$. Given that
    $$a_0 + a_1 + a_2 + \cdots + a_{2024} = 2024,$$
    compute a_2.

12. Let N_{11} be the **smallest prime that does not divide** the answer to Problem 11.

    In the diagram below, let G be the intersection of $\overline{AB}$ and $\overline{DE}$, and let H be the intersection of $\overline{AC}$ and $\overline{DF}$. Suppose that equilateral triangle $\triangle ABC$ is the reflection of triangle $\triangle DEF$ about the line $\overleftrightarrow{GH}$ and that $AB = N_{11} \cdot AG$. Let circle O_1 be tangent to all four sides of quadrilateral $AGDH$, and let points B, C, F, and E all lie on circle O_2. Find the ratio of the area of O_1 to the area of O_2. Note that the diagram below is not drawn to scale.



Set 5



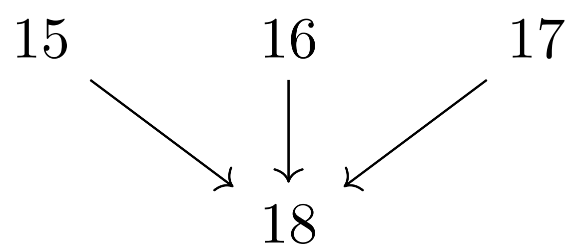
13. Let N_{14} be the answer to Problem 14.

    The lines $y = \frac{1}{20}x$ and $y = \frac{1}{20-N_{14}}x$ intersect the line $y = \frac{\sqrt{2}}{2}$ at distinct points A and B, respectively. Let point O be the origin of the coordinate plane. Find the area of triangle $\triangle ABO$, given that it is a positive integer.

14. Let N_{13} be the answer to Problem 13.

    Find the number of ways to choose two positive integers a and b such that $a \leq b < a+b \leq 2N_{13}$.

Set 6



15. Justin is playing a game. He initially writes the following list of 7 numbers:

$$1, 1, 1, 2, 3, 5, 8.$$

Then, on every turn of the game, he chooses either the median (the 4^{th} greatest element) or arithmetic mean (the sum of all elements divided by 7) of the list. He adds that number to every element of the list, and this becomes his new list. If the arithmetic mean of Justin's list is greater than 500, the game ends. What is the minimum number of turns it could take for the game to end?

16. For a non-negative integer a and a positive integer $b \geq 2$, let $a \star b$ denote the remainder when $a + 1$ is divided by b. For example, $2 \star 3 = 0$ and $1 \star 10 = 2$. Compute

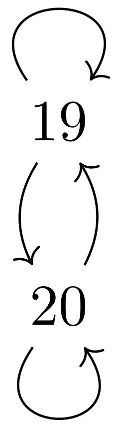
$$(((((1 \star 2024) \star 2023) \star 2022) \star \cdots) \star 61).$$

17. Zain is riding a bike in the coordinate plane. They start at $(0, 0)$ and bike to all the points with integer coordinates in a spiral pattern: they first bike to $(0, 1)$, then to $(-1, 1)$, $(-1, 0)$, $(-1, -1)$, $(0, -1)$, and so on, turning left if they haven't visited the point to their left and going straight otherwise. Every time Zain reaches a point (x, y) whose coordinates satisfy $|x + y| = 2024$, they briefly celebrate on that point. Suppose the 4144^{th} point Zain celebrates on is (a, b). Compute b.

18. Let N_{15} be the answer to Problem 15, N_{16} be the answer to Problem 16, and N_{17} be the answer to Problem 17.

Triangle $\triangle XYZ$ has a right angle at Y. Points A and D lie on $\overline{XY}$ and $\overline{YZ}$, respectively, such that $\overline{AD}$ is parallel to $\overline{XZ}$. Let $\overline{AD}$ intersect the inscribed circle of $\triangle XYZ$ at points B and C, with A closer to B than C. Suppose $AB = N_{16}$, $BC = N_{17}$, and $CD = N_{15}$. Compute the smallest possible value of XZ.

Set 7



19. Let N_{19} be the answer to this problem and N_{20} be the answer to Problem 20.

    Parallel lines ℓ_1, ℓ_2, and ℓ_3 are arranged with ℓ_2 between ℓ_1 and ℓ_3 such that adjacent lines are distance N_{20} apart. Point A lies on ℓ_1, point B lies on ℓ_2, and point C lies on ℓ_3 such that $AB = N_{19}$ and $BC = N_{20}\sqrt{10}$. Compute AC.

20. Let N_{19} be the answer to Problem 19 and N_{20} be the answer to this problem.

    Suppose that exactly 2 of the following 3 expressions are equal:

    $$N_{20} + N_{19},\ 2N_{20} - N_{19},\ \text{and}\ N_{19}N_{20}.$$

    Compute N_{20}.