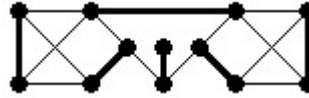1. (a) Unique solution:

(b) Let $P(G)$ denote the number of perfect matchings on graph G. Label the vertices $1, ..., 2n$; then we note that vertex 1 can be matched with any of the remaining $2n - 1$ vertices. Deleting other edges connected to these vertices gives us a K_{2n-2}; thus, $P(K_{2n}) = (2n - 1)P(K_{2n-2})$.

We claim that $P(K_{2n}) = (2n - 1)(2n - 3)...1$. To prove this, we use induction. First, our base case is $P(K_2) = 1$. Suppose it's true for $P(K_{2n-2})$. Then for K_{2n}, $P(K_{2n}) = (2n - 1)P(K_{2n-2}) = (2n - 1)(2n - 3)...1$ by the inductive hypothesis.

When $n = 6$, we have $11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 10395$.

(c) We split into cases: odd n and even n

- Case 1: n is odd.
  Consider the center polygon. If an edge is matched in that case, the corresponding outside polygon will not have a valid matching since the number of sides is odd. Thus, we can remove all center edges. We are then left with a cycle with an even number of vertices, so there are exactly 2 perfect matchings in this case.

- Case 2: n is even.
  This time, the center polygon can have edges matched. Look at the center polygon only. We can label the edges alternating on or off. If an edge is on, then there are two perfect matchings corresponding to the outside polygon. If an edge is off, then there is only one perfect matching. There are $\frac{n}{2}$ on edges in the center polygon, and there are two ways to orient each of them, and there are two ways to assign the on and off edges in the center polygon, thus the answer is $2 \cdot 2^{\frac{n}{2}} = 2^{\frac{n}{2}+1}$

(d) Color the vertices as follows:
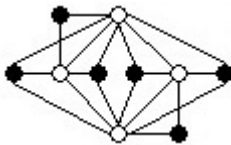
We notice that each edge contains at least 1 white vertex. However, there are only 4 white vertices and 6 black vertices, so there cannot be a perfect matching, as every one of the 5 edges in the matching must touch a white vertex.

2. (a) Let G_n denote this graph.
   We use induction to show that the number of perfect matchings is F_{n+2}, where F_i denotes the ith Fibonacci number.
   Consider the upper left vertex. We have two cases:
   First, the matching contains the vertical edge touching this vertex. Then we remove all edges associated with the left two vertices, leaving us with a copy of G_{n-1} which can be matched as such. Second, the matching contains the horizontal edge touching this vertex. Then after removing this edge, the bottom left vertex has only the horizontal edge sticking out of it and thus that edge is required to be in the matching. After removing it as well, we are left with a copy of G_{n-2}.
   Since these are the two cases, we have $G_n = G_{n-1} + G_{n-2}$. Now, we notice that $G_0 = 1$ and $G_1 = 2$; thus, the unique G_n stemming from such a recurrence must be $G_n = F_{n+2}$.

(b) We claim that there are $2F_nF_{n+1}$ matchings. To show this, first, we check to see that this is true for the base cases of $n = 1$ and $n = 2$. We consider where the vertex highlighted in the below graph goes to. Note that the graph is symmetric.
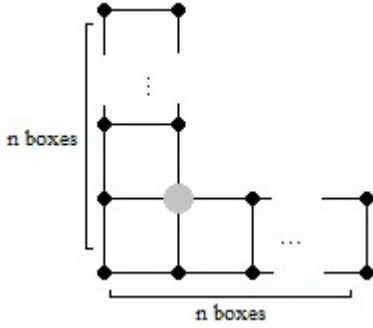
In the first case, we have the vertex matched by the vertical edge leading up, as seen below. Note that this means the gray edge highlihgted below is forced, as otherwise the gray vertex will be forced to match with the dashed line in the below graph. This would disconnect the graph into two connected components, each with an odd number of vertices, leaving them unable to be matched.
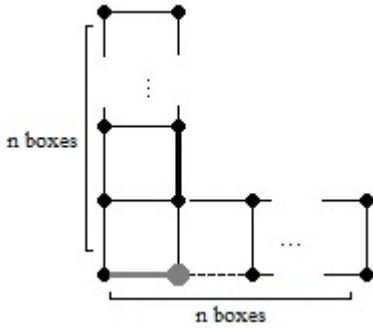

This leads us to the following diagram of forced and forbidden edges (bold edges are forced, crossed-out edges are forbidden):
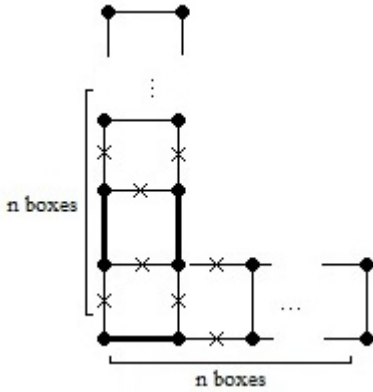
which would give us $F_{n-1}F_n$ matchings by (a), since the number of matching of a graph consisting of two disjoint pieces is the product of the number of matchings on each piece.

In the second case, we have the horizontal edge leading left, as seen below. This forces the horizontal edge shown in gray, and leads to the following diagram of forced and forbidden matches:
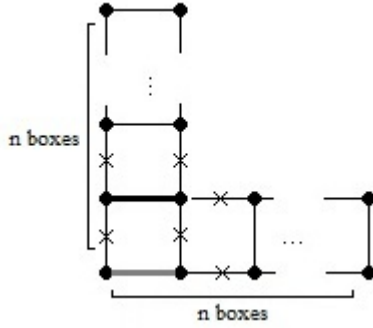
which would give us F_n^2 matchings by (a).

By symmetry, we get the same for the other horizontal edge and the other vertical edge leading from our chosen vertex. This gives us a total of

$$2F_{n-1}F_n + 2F_n^2 = 2F_n(F_{n-1} + F_n) = 2F_nF_{n+1}$$

matchings.

3. Every forest has at least 1 leaf - a vertex with degree 1. (This can be shown using pigeonhole - no cycles means the graph, with n vertices, has $\leq n-1$ edges.) Then we use induction. A forest of size 2 has either one perfect matching or none. Suppose every forest of $n-1$ size has at most 1 perfect matching. Then we have our forest of size n, pick a leaf and the unique vertex adjacent to it. If there is a matching, this edge will be forced so we can remove it from consideration. By induction, the rest of the graph has at most 1 perfect matching, and thus the forest has at most 1 perfect matching.

4. We claim that G has a *Hamiltonian cycle*, that is, a cycle that includes all points in G exactly once.

Suppose that we have a graph G satisfying these conditions that does not have a Hamiltonian cycle. Then we can find one in the set of all such G with the maximum number of edges. This graph, call it H, must contain a path $\{v_1, ..., v_{2n}\}$ that contains all points in H, otherwise we may add more edges without creating a Hamiltonian cycle.

v_1 is not adjacent to v_{2n} because H does not have a Hamiltonian cycle. We notice that $\deg(v_1)+\deg(v_{2n}) \geq 2n$. Thus by pigeonhole, we have some i so that v_i is adjacent to v_1 and v_{i-1} is adjacent to v_{2n}. Then

$$\{v_1, ..., v_{i-1}, v_n, ..., v_i, v_1\}$$

is a Hamiltonian cycle, reaching a contradiction.

Now that G has a Hamiltonian cycle, we simply take every other edge of the cycle to obtain our desired perfect matching.

5. For the easier part, we prove that if a graph has a perfect matching, then $|\text{adj}(S)| \geq |S|$. Let M be a perfect matching. For every set $S \subseteq V_1$, for each vertex $v \in S$, consider the vertex v' that v is matched to M. Since all v' are distinct, and the set of all v' is a subset of the set of all neighbors to v, the result follows. $\square$

For the other direction, we must prove that $\forall S \subseteq V_1, |\text{adj}(S)| \geq |S|$ implies that G has a perfect matching. First, we define a few terms:

- Define an edge to be "on" if it is part of the current matching, and off otherwise.
- Define an "augmenting path" to be a path through the graph that starts with an off edge and alternates between on and off edges.

We proceed by contradiction. Suppose that G does not have a perfect matching, but instead, has a maximum matching M. Consider an unmatched vertex v, and consider all augmenting paths coming out of it.

Lemma 1: All augmenting paths from v end on a vertex in V_1

Proof of 1: Suppose that the augmenting path ended on a vertex in V_2. Then, we could take that path and toggle the state of each edge, from on to off and vice versa. This is still a valid matching, since all vertices have at most one on edge incident from it. However, since the path starts and ends on an off edge, this will lead to matching with size $|M| + 1$, which contradicts the fact that M is a maximal matching, so we are done. $\square$

Let A the set of all vertices in V_1 that is connected to v by some augmenting path, and let B be the set

of all vertices in V_2 that is connected to v by some augmenting path. Consider the set C which is all the vertices in V_2 that is matched to some vertex in A in M.

Lemma 2: Except for v, all vertices in A are matched in M.

Proof of 2: Since we start with an off edge, we enter a vertex in V_2 using an off edge, and we enter a vertex in V_1 using an on edge. Since we must enter using an on edge, all vertices in A except for v are matched in M. $\square$

Since all vertices in A are matched except for v, we must have $|C| = |A| - 1$.

Lemma 3: All vertices in B are matched in M.

Proof of 3: Using lemma 1, all vertices in B must have an on edge directing it to a vertex in V_1, which shows that all vertices in B are matched. $\square$

That must mean $|B| = |C|$ as well.

Lemma 4: $\text{adj}(A) = B$

Proof of 4: Whenever an augmenting path reaches a vertex in V_1, it must have entered using an on edge. The augmenting path can be then increased by exhausting all other edges of V_1, which all must be off (since there is at most one on edge). Thus, all neighbors of A are in B.

Using all the lemmas above, we have $|\text{adj}(A)| = |B| = |C| = |A| - 1$, which implies $|\text{adj}(A)| < |A|$, which is the contradiction that we are looking for. $\square$

6. We claim that $x_n = F_{2n-1}$ when $n \geq 1$. To prove this, we use induction. Notice this is true for $n = 1, 2$. Suppose it is true for $n - 1$. Then for n,

$$x_n = \frac{x_{n-1}^2 + 1}{x_{n-2}} = \frac{F_{2n-3}^2 + 1}{F_{2n-5}}$$

$$= \frac{(F_{2n-4} + F_{2n-5})^2 + 1}{F_{2n-5}} = \frac{(F_{2n-4})(F_{2n-4} + 2F_{2n-5}) + F_{2n-5}^2 + 1}{F_{2n-5}}$$

$$= \frac{(F_{2n-5} + F_{2n-6})(F_{2n-3} + F_{2n-5}) + F_{2n-7}F_{2n-3}}{F_{2n-5}}$$

$$= \frac{(F_{2n-5})(F_{2n-3} + F_{2n-5} + F_{2n-6}) + F_{2n-6}F_{2n-3} + F_{2n-7}F_{2n-3}}{F_{2n-5}}$$

$$= \frac{(F_{2n-5})(F_{2n-2}) + F_{2n-5}F_{2n-3}}{F_{2n-5}}$$

$$= F_{2n-1}.$$

7. 2,3,7,23, (59,...)

8. We use strong induction. We notice that $a_0, ..., a_7$ are integral by the previous problem. Now, suppose that $a_0, ..., a_{n-1}$ are all integers.

First, we notice that a_k and a_{k-1} are relatively prime for all $k < n$, since otherwise if $p|a_k$ and $p|a_{k-1}$, then by the recurrence, $p|a_{k-2},..., p|a_3 = 1$.

From this, we see a_k and $a_{k-2}a_{k-3}$ are relatively prime for all $k < n - 1$. For if $p|a_k$, then p cannot divide a_{k-3}, otherwise $a_{k-3}a_{k+1} = a_{k-1}^2 + a_{k-2}a_k$ implies $p|a_{k-1}$. Also, p cannot divide a_{k-2} because $a_k a_{k-4} = a_{k-1}a_{k-3} + a_{k-2}^2$ would then give us $p|a_{k-1}a_{k-3}$.

Now, we claim that $a_{n-3}a_{n-1} + a_{n-2}^2 \equiv 0 (\mod a_{n-4})$. This will show that a_n is an integer, as desired. We have that

$$a_{n-7}a_{n-6}a_{n-3}a_{n-1} + a_{n-7}a_{n-6}a_{n-2}^2$$

$$\equiv a_{n-6}a_{n-5}^2 a_{n-1} + a_{n-7}a_{n-6}a_{n-2}^2$$

$$\equiv a_{n-6}a_{n-5}a_{n-3}^2 + a_{n-7}a_{n-6}a_{n-2}^2$$

$$\equiv a_{n-6}a_{n-5}a_{n-3}^2 + a_{n-7}a_{n-5}a_{n-3}a_{n-2}$$

$$\equiv a_{n-6}a_{n-6}a_{n-2}a_{n-3} + a_{n-7}a_{n-5}a_{n-3}a_{n-2}$$

$$= a_{n-3}a_{n-2}(a_{n-6}a_{n-6} + a_{n-7}a_{n-5})$$

$$= a_{n-3}a_{n-2}(a_{n-8}a_{n-4}) \equiv 0(\mod a_{n-4}).$$

Thus $\frac{a_{n-3}a_{n-1} + a_{n-2}^2}{a_{n-4}} = a_n$ is an integer, as desired. This completes the induction and the proof.

9. Let $a_i = \frac{n_i}{d_i}$, where n_i and d_i are integers and $\gcd(n_i, d_i) = 1$. We claim that

$$d_m \mid \gcd(n_2, n_3, n_4) \prod_{i=0}^{m-1} d_i.$$

Note that this holds trivially when $m \leq 7$; we now proceed by induction.

First, we will show that if p does not divide $\prod_{i=0}^{m-1} d_i$ but $p \mid n_m$ and n_{m-1}, then $p \mid n_k$ for any $1 < k < m$. This follows from the recurrence relation $a_k a_{k-4} = a_{k-1} a_{k-3} + a_{k-2}^2$ which gives us

$$n_m = \frac{n_{m-1} n_{m-3} d_m d_{m-2}^2 d_{m-4} + n_{m-2}^2 d_m d_{m-1} d_{m-3} d_{m-4}}{n_{m-4} d_{n-1} d_{n-2}^2 d_{n-3}} (*)$$

from which we see that $p \mid n_{m-2}$. Continuing downwards, we eventually get $p \mid n_k$ for all $1 < k < m$.

Now, we claim that if $m > 7$, $p \mid d_m$ but p does not divide $\prod_{i=0}^{m-1} d_i$, then $p \mid \gcd(n_{m-7} n_{m-6}, n_{m-4})$.
We have

$$a_{m-7} a_{m-6} a_{m-4} a_m$$
$$= a_{m-7} a_{m-6} a_{m-3} a_{m-1} + a_{m-7} a_{m-6} a_{m-2}^2$$
$$= a_{m-6} (a_{m-5}^2 + a_{m-4} a_{m-6}) a_{m-1} + a_{m-7} a_{m-6} a_{m-2}^2$$
$$= a_{m-6} a_{m-5} (a_{m-3}^2 + a_{m-2} a_{m-4}) + a_{m-6}^2 a_{m-4} a_{m-1} + a_{m-7} a_{m-6} a_{m-2}^2$$
$$= a_{m-6} a_{m-5} (a_{m-3}^2 + a_{m-2} a_{m-4}) + a_{m-6}^2 a_{m-4} a_{m-1} + a_{m-7} (a_{m-5} a_{m-3} + a_{m-4}^2) a_{m-2}$$
$$= a_{m-6} (a_{m-6} a_{m-2} - a_{m-4}^2) a_{m-3} + a_{m-6} a_{m-5} a_{m-2} a_{m-4}$$
$$+ a_{m-6}^2 a_{m-4} a_{m-1} + a_{m-7} (a_{m-5} a_{m-3} + a_{m-4}^2) a_{m-2}$$
$$= a_{m-6} a_{m-6} a_{m-2} a_{m-3} - a_{m-6} a_{m-4}^2 a_{m-3}$$
$$+ a_{m-6} a_{m-5} a_{m-2} a_{m-4} + a_{m-6}^2 a_{m-4} a_{m-1} + a_{m-7} a_{m-5} a_{m-3} a_{m-2} + a_{m-7} a_{m-4}^2 a_{m-2}$$
$$= (a_{m-6} a_{m-6} + a_{m-7} a_{m-5}) a_{m-2} a_{m-3} - a_{m-6} a_{m-4}^2 a_{m-3}$$
$$+ a_{m-6} a_{m-5} a_{m-2} a_{m-4} + a_{m-6}^2 a_{m-4} a_{m-1} + a_{m-7} a_{m-4}^2 a_{m-2}$$
$$= a_{m-8} a_{m-4} a_{m-2} a_{m-3} - a_{m-6} a_{m-4}^2 a_{m-3}$$
$$+ a_{m-6} a_{m-5} a_{m-2} a_{m-4} + a_{m-6}^2 a_{m-4} a_{m-1} + a_{m-7} a_{m-4}^2 a_{m-2}$$
$$= a_{m-4} (a_{m-8} a_{m-2} a_{m-3} - a_{m-6} a_{m-4} a_{m-3}$$
$$+ a_{m-6} a_{m-5} a_{m-2} + a_{m-6}^2 a_{m-1} + a_{m-7} a_{m-4} a_{m-2}).$$

Thus if $p \mid d_m$, then $p \mid n_{m-7} n_{m-6} \prod_{i=0}^{m-1} d_i$.
Meanwhile, from $a_{m-4} a_m = a_{m-3} a_{m-1} + a_{m-2}^2$, if $p \mid d_m$ then $p \mid n_{m-4} d_{m-3} d_{m-2} d_{m-1}$. Thus if p does not divide $\prod_{i=0}^{m-1} d_i$, then $p \mid \gcd(n_{m-7} n_{m-6}, n_{m-4})$.
Finally, we shall show that if $p \mid \gcd(n_{m-7} n_{m-6}, n_{m-4})$ and p does not divide $\prod_{i=0}^{m-1} d_i$, then $p \mid n_{m-5}$.
In the first case, suppose $p \mid n_{m-7}$. Then plugging in $m-3$ to (*), we get $p \mid n_{m-5}$.
In the second case, suppose p does not divide n_{m-7}; then $p \mid n_{m-6}$. Then plugging in $m-4$ to (*), we get $p \mid n_{m-5} n_{m-7}$ and thus $p \mid n_{m-5}$. Thus, if $p \mid d_m$ does not divide $\prod_{i=0}^{m-1} d_i$, then $p \mid \gcd(n_2, n_3, n_4)$.

10. Let a_n denote the number of matchings for the following graph:

2n-2 boxes
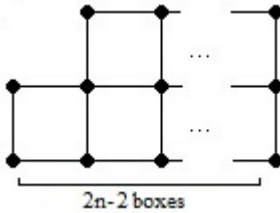
Then by looking at the edge matching the middle left point of g_n, we obtain

$$g_n = 2a_n + g_{n-1}.$$

Meanwhile, by looking at the edge matching the bottom left point of a_n, we obtain

$$a_n = a_{n-1} + g_{n-1}.$$

From these, we deduce

$$g_n - g_{n-1} = 2a_n$$

5

and

$$a_n - a_{n-1} = g_{n-1}.$$

Thus

$$g_{n-1} = \frac{g_n - g_{n-1}}{2} - \frac{g_{n-1} - g_{n-2}}{2} = \frac{g_n}{2} + \frac{g_{n-2}}{2} - g_{n-1}$$

so

$$4g_{n-1} = g_n + g_{n-2}.$$

Now, we use induction to show that $g_n g_{n-2} = g_{n-1}^2 + 2$. We see that it is true for $g_0 = 1, g_1 = 3, g_2 = 11$. Suppose $g_{n-1} g_{n-3} = g_{n-2}^2 + 2$. Then

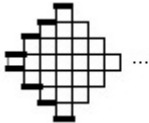$$g_n g_{n-2} = (4g_{n-1} - g_{n-2})g_{n-2}$$

$$= 4g_{n-1}g_{n-2} - g_{n-2}^2$$

$$= 4g_{n-1}g_{n-2} - (g_{n-1}g_{n-3} - 2)$$

$$= g_{n-1}(4g_{n-2} - g_{n-3}) + 2$$

$$= g_{n-1}^2 + 2.$$

11. Suppose the perfect matching does not contain the gray westmost edge shown in the following diagram.

Then the two leftmost horizontal edges are both forced to be in the matching. By eliminating edges touching vertices already matched and noticing that a vertex of degree 1 has that edge forced to be in the matching, we continue to see that all the bold edges in hte following diagram are forced to be in the matching.

In particular, the topmost and bottommost edges are in the matching.

12. (a) Let A_n be the number of matchings of the nth Aztec Diamond. We claim that

$$A_n A_{n-2} = 2A_{n-1}^2.$$

To prove this, we show that the number of ordered pairs (A, B) is twice the number of ordered pairs (C, D), where A, B, C, D are matchings of the $n, n-2, n-1, n-1$th Aztec diamonds, respectively. We superimpose a matching B of an $n-2$-diamond on top of a matching A of an n-diamond as shown below, so that the smaller diamond is centered inside the larger one.
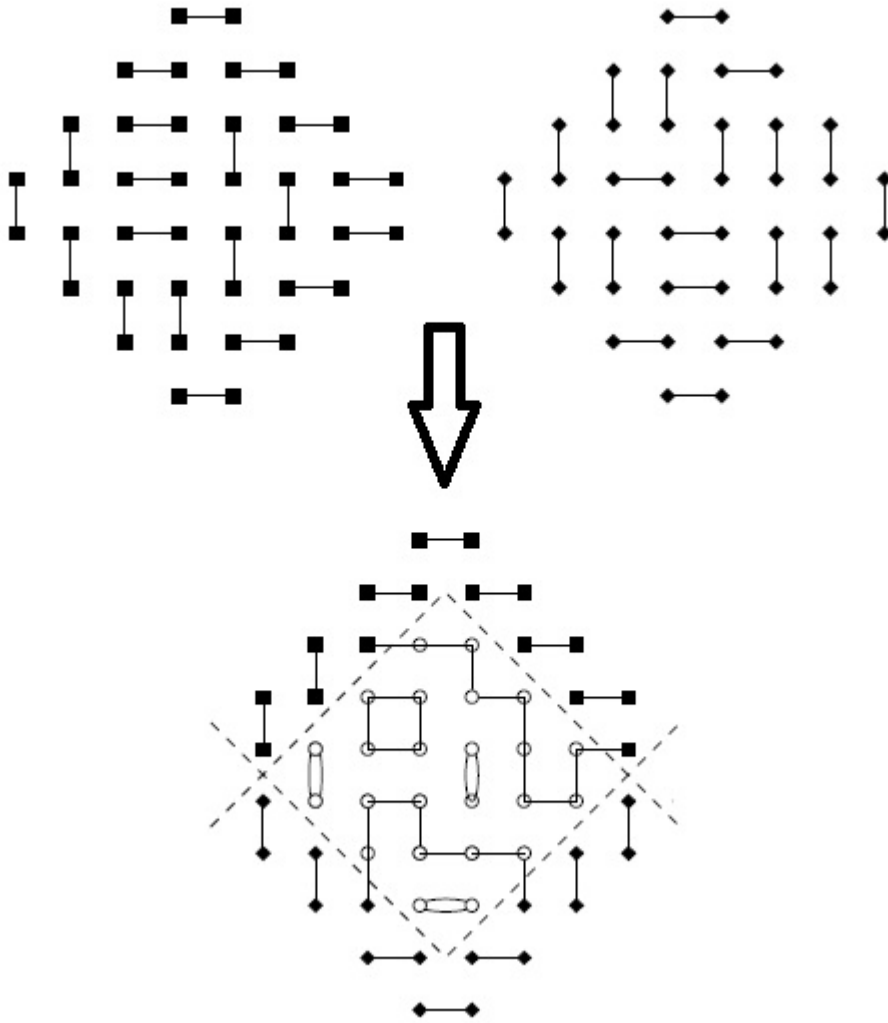
6
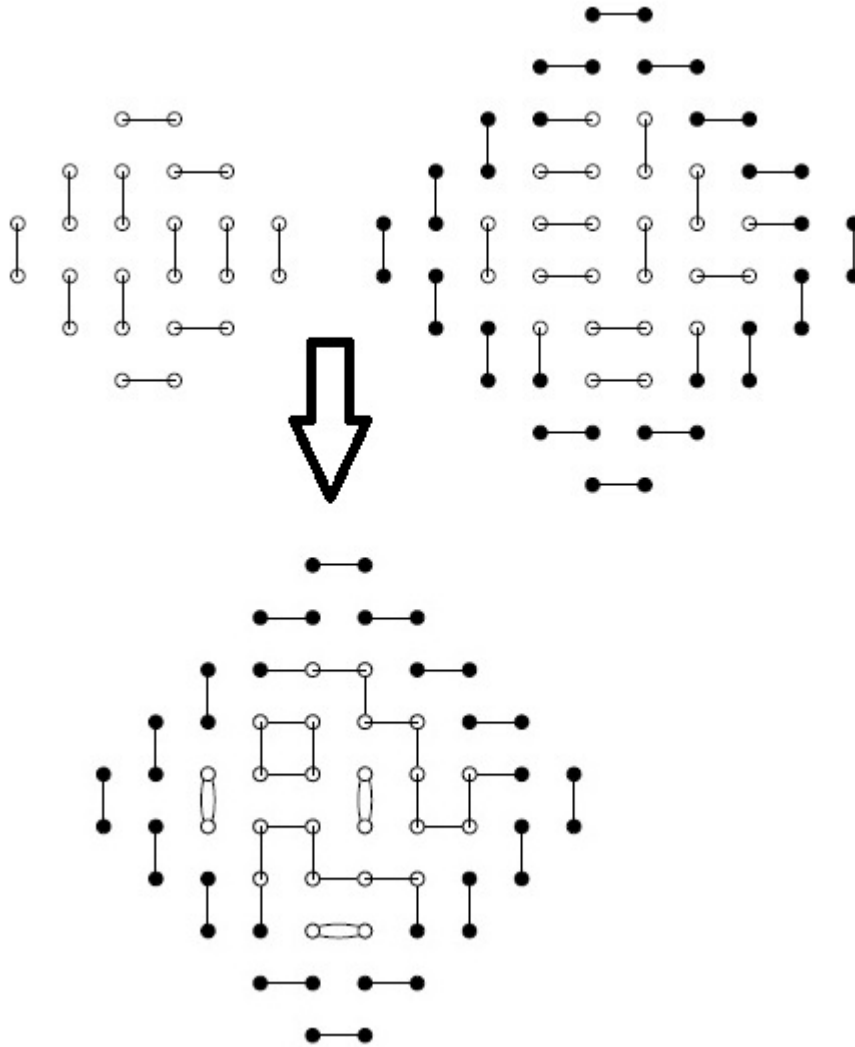
We may also overlap two matchings C, D of $n-1$-diamonds so that the overlapping region is a $n-2$-diamond, as shown here:

This may also be done vertically - there are two possible ways to fit together two such matchings in this fashion. In either case, we may add the two "'missing"' edges naturally to form a multigraph on the vertices of the n-diamond, where all vertices in the central $n-2$-diamond have degree 2, and the other vertices have degree 1.

We claim that the graphs G formed in this fashion are the same as those formed by superimposing a matching B of an $n-2$-diamond on top of a matching A of an n-diamond.

We note that G is bipartite, and thus all cycles have even length. These cycles are contained in the middle $n-2$ diamond, whose vertices have degree 2. Each cycle can then be partitioned so that every other edge goes to the same subgraph; adjacent edges go to different subgraphs. For each cycle, there are two ways to decide which half of the cycle goes to A or B. Similarly, there are two ways to decide which half goes to C or D. All doubled edges in G are split and shared by each subgraph. It remains to show that the other edges must be partitioned uniquely.

In the graph below, each black vertex must be joined to a gray vertex.
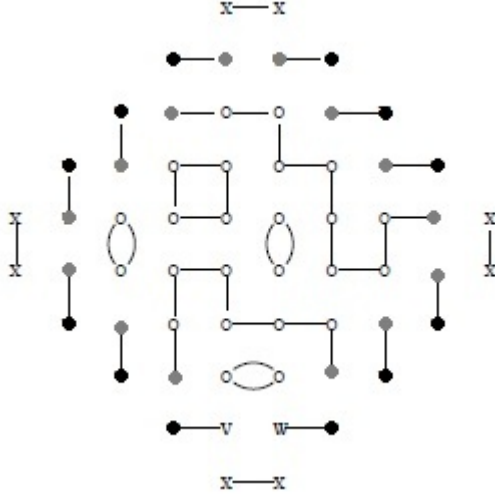
Thus, on each side of the diamond, we have exactly one gray vertex not connected to a black vertex. There are three options for this lonely gray vertex:

1. It connects to a white vertex inside the $n-2$-diamond.

2. It connects to another lonely gray vertex.

2. It connects to one of the x-vertices (which forces the other x-vertex in that pair to connect to a lonely gray vertex on the neighboring side).

In all these cases, we have a path between two lonely gray vertices. We note that any such path connecting two lonely gray vertices must have the gray vertices be from adjacent edges, because otherwise the two paths formed by the four lonely gray vertices would cross in the middle of the white vertices. Since the white vertices have degree 2, this cannot happen.

Now, we claim that both end edges in such a path will belong to the same subgraph in a partition of G. This is easily shown by coloring the vertices of G in a checkerboard pattern. The two end vertices are then different colors, and thus the path between them has an odd number of edges. Thus the two end edges are in the same subgraph.

When partitioning G into a n-diamond and a $n-2$ diamond, we simply let all four such edges be part of A and determine the rest thereafter. Such a partition always exists.

When partitioning G into two $n-1$-diamonds, we let each diamond take the two such edges that are connected by a path. This forces the direction the diamonds overlap in (left-to-right or up-and-down) and the rest of the partitioning is straightforward. Hence each doubled Aztec graph can be partitioned into two order-$(n-1)$ Aztec matchings in one way (top-bottom) or the other (left-right), but never both. The partition of the paths is uniquely determined. The number of ways to combine Aztec matchings of orders n and $n-2$ is $A_n A_{n-2}$, whereas the number of ways to combine two order-$n-1$ matchings is $2A_{n-1}^2$. Thus $A_n A_{n-2} = 2A_{n-1}^2$ and we are done.

(b) We claim $A_n = 2^{\frac{n(n+1)}{2}}$. We prove this using induction. First we note that $A_1 = 2$ and $A_2 = 8$. Suppose it's true for $k < n$. Then for n,

$$A_n = \frac{2A_{n-1}^2}{A_{n-2}} = \frac{2 \cdot 2^{\frac{n(n-1)}{2} \cdot 2}}{2^{\frac{(n-1)(n-2)}{2}}}$$

$$= 2^{n(n-1)+1-\frac{(n-1)(n-2)}{2}} = 2^{\frac{n(n+1)}{2}}.$$